M.Sc. 1st Semester Examination-2022-23

MATHEMATICS

Course ID : 12152 **Course Code : MATH/102C**

Course Title : Linear Algebra & Module Theory

Time : 2 Hours *Full Marks* : 40

The figures in the right hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer **any *five*** questions : $8\times5=40$

1. (a) Let V and W be n-dimensional vector spaces over the field F and $T : V \to W$ be a linear transformation. Prove that T is linear if and only if the null space $N(T)=\{0_V\}$.

   (b) Consider the basis $B = \{(1, 1, -1), (-1, 1, 1), (1, -1, 1)\}$ for the vector space R^3 over R. Find the dual basis of B.

(c) Let V be the vector space of all nxn matrices over R. Let B be fixed nxn matrix. If T is a linear operator on V defined by T(A) = AB – BA, and f is the trace function, what is $D^t f$? 2+4+2

2. (a) Find the eigenvalues and eigenspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

Is A diagonalizable? Give reasons for your answer.

(b) Write the vector space R^3 over R as the direct sum of two of its subspaces. (2+2+2)+2

3. (a) State and proof Cayley-Hamilton theorem for a linear operator.

(b) Define a triangulable linear operator. What can you say about the minimal polynomial of triangulable linear operator? (2+4)+(1+1)

4. (a) Define $T : R^3 \to R^3$ by $T(x, y, z) = (-x, x - y, 3x + 2y + z)$. Check whether T satisfies the polynomial $(x - 1)(x + 1)^2$. Find the minimal polynomial of T.

(b) Find the rational canonical form of the matrix

$$A = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}$$

(2+2)+4

5. (a) Let P_2 be the vector space with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Show that the constant function 1 and the identity function are orthogonal in P_2.

(b) Let V be a finite dimensional inner product space over F and let $g : V \to F$ be a linear transformation. Then prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

(c) Prove that for a normal operator T on an inner product space V, $||Tv|| = ||T^*v||$ for all $v \in V$.

2+4+2

6. (a) Use Gram-Schmidt process to obtain an orthonormal basis from the basis set {(3, 0, 4), (-1, 0, 7), (2, 9, 11)} of the Euclidean space R^3 with standard inner product.

(b) Consider the bilinear form $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H((a_1, a_2), (b_1, b_2)) = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Find the matrix representation of H w.r.t. the ordered basis $B = \{(1, 2), (2, 3)\}$ of $\mathbb{R}^2$. 6+2

7. (a) Define a left R-module over the ring R.

(b) Define the annihilator of an R-module. If M is an R-module annihilated by an ideal I, then prove that M is an (R/I)-module. 2+(2+4)

8. (a) If N and K are left R-modules of M then show that $N \cap K$ is a submodule of M. Give example to show that union of two submodules may not be a submodule.

(b) Is a linear transformation between vector spaces a R-module homomorphism ?

(c) State the structure theorem for finitely generated modules over a PID (Principal ideal domain).

(3+2)+1+2

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